Substituting (16) into (17) gives

$$t_2 - t_1 = \int_{v_1}^{v_2} F(p, v)dv \tag{18}$$

where

$$F(p, v) = \left(\frac{p^3}{\mu}\right)^{1/2} \frac{1}{(1 + K_1 \, p^{1/2} \cos v)^2} \tag{19}$$

In order that the transfer time given by Eq. (18) increase (or decrease) with $p$, it is sufficient that $F(p, v)$ increase (or decrease) monotonically with $p$ for all values of $v$ in the interval considered. Differentiating (19) with respect to $p$ and remembering Eq. (16) yields

$$\frac{\partial F(p, v)}{\partial p} = \frac{1}{2}\left(\frac{p}{\mu}\right)^{1/2} \frac{3 + e \cos v}{(1 + e \cos v)^3} \tag{20}$$

which is always positive for elliptical orbits.

Hence, it is shown that the time interval increases monotonically with $p$. Thus, if the initial estimate for $p$ yields a time of flight greater (less) than the observed time increment, the second estimate is lowered (raised) relative to the first. The foregoing procedure may be used to obtain values of $p$ yielding times of flight bounding the observed increment. These values then may be used as initial estimates for the iteration procedure. The foregoing constitutes a proof of the uniqueness of the solution, i.e., each $p$ is associated with one and only one time of flight.

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# Resonance Scattering Photography of Free Molecular Flow

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An experiment is described in which the free molecular flow of sodium vapor from an orifice was photographed, using the resonance scattering technique. The Knudsen number in the upstream chamber based on the orifice diameter was about 40. Photographs are presented of the flow over a cylinder and a plate.

### Introduction

THE resonance scattering technique for visualizing flow makes it possible to obtain photographs at density levels several orders of magnitude lower than with previous techniques.[1] Application of this technique to a simple free molecular flow in which the properties of the flow field are known is desirable. The effusion of gas from a container into vacuum through an orifice is such a flow. The properties of free jet issuing from the orifice may be calculated quite accurately from simple kinetic theory considerations.[2] Another advantage of this flow is that, simply by changing the upstream density, the jet flow may be varied from continuum to free molecule. This note describes an experiment that was devised to illustrate the use of resonance scattering for visualizing free molecular orifice flow.

### Theory of Resonance Scattering

The present method for observing low density gas flow (number of atoms per cubic centimeter between $10^7$ and $10^{12}$) is based on the knowledge that the scattering cross section of photons close to resonance becomes very large. The intensity of scattered radiation compared to the incident intensity is[3]

$$I/I_0 = \phi(\theta)/R^2 \tag{1}$$

where $I$ and $I_0$ are the scattered and incident intensities, $R$ is the distance to the scattering center, and $\phi$, for classical elastically bound electrons, is

$$\phi(\theta) = \frac{e^2}{mc^2} \frac{\nu^4 \sin^2(\theta)}{(\nu_0^2 - \nu^2)^2 + \nu^2\gamma^2} \tag{2}$$

Here $e$ is electronic charge, $m$ electron mass, $c$ velocity of light, $\nu$ frequency of incident light, $\nu_0$ frequency of resonance light, and

$$\gamma = 2e^2\nu_0^2/3mc^3 \tag{3}$$

Thus the total classical scattering cross section can be obtained by integrating Eq. (2) over a sphere:

$$\phi_{\text{total}} = \frac{8\pi}{3} \frac{e^4}{m^2c^4} \frac{\nu^4}{(\nu_0^2 - \nu^2)^2 + \nu^2\gamma^2} \tag{4}$$

This can be approximated for the immediate neighborhood of resonance as

$$\phi_{\text{total}} = \frac{2\pi e^4}{3m^2c^4} \frac{\nu^2}{(\nu_0 - \nu)^2 + (\gamma^2/4)} \tag{5}$$

Quantum mechanical treatment of the scattering process introduces only a moderate correction. Thus, it is found that the total cross section of light exactly at resonance is

$$\phi_{\text{total}} = \lambda^2/2\pi \tag{6}$$

where $\lambda$ is the wavelength of the resonance light. For the present application of sodium $D$ lines to low density and low temperature sodium vapor, the scattering cross section is

$$\phi_{\text{total}} \approx (6 \times 10^{-5})^2/6 = 6 \times 10^{-10} \text{ cm}^2$$

Therefore a density of sodium atoms of $10^9$ cm$^{-3}$ will scatter a large fraction of the incident light. Densities of the order of $10^7$ particles/cm$^3$ should be observable by the resonance scattering method.

### Free Molecular Effusion

Consider the free molecular effusion of sodium atoms through a small orifice of diameter $d$ and negligible lip thickness. The flow may be considered free molecular if the mean free path $\Lambda$ in the container is large compared to $d$. The number of atoms per unit time escaping the orifice is therefore

$$N_i = \tfrac{1}{4}\bar{C}N_0A \tag{7}$$

where

$\bar{C} = (3KT_0/m)^{1/2} =$ mean molecular velocity upstream of the orifice

$N_0 =$ number of atoms per unit volume in the source

$T_0 =$ temperature of the source
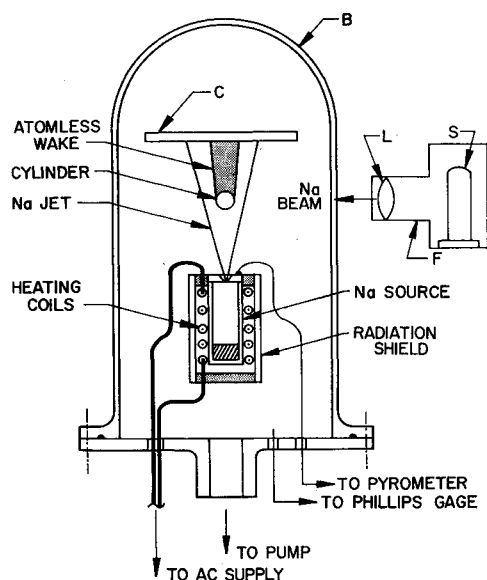
Fig. 1 Experimental apparatus



a) Free jet



b) $\frac{1}{8}$-in.-diam cylinder spanning the jet



c) $\frac{1}{2}$-in.-wide plate spanning the jet

Fig. 2 Resonance scattering photographs of the free molecular jet: ambient pressure $10^{-5}$ mm Hg, source temperature 300°C, and source pressure $10^{-2}$ mm Hg

$A$ = area of the orifice
$K$ = Boltzmann's constant
$m$ = mass of Na atom

The mean free path $\Lambda_0$ upstream of the orifice is given in terms of the diameter $d$ of the Na atoms by

$$\Lambda_0 = 2^{-1/2}(N_0 \pi d^2)^{-1} \qquad (8)$$

The Knudsen number based on the orifice diameter and the mean free path $\Lambda_0$ is

$$K_0 = \Lambda_0/d \qquad (9)$$

After passing through the orifice, the atoms move in straight lines with a negligible number of collisions provided the mean free path for scattering by air molecules in the downstream reservoir is sufficiently large. The half angle $\theta$ that the jet makes with the source center line is determined by the geometry of the orifice. Once the half angle is known, the radius of the jet at any distance $l$ from the orifice is given by

$$r_j = l \tan\theta \qquad (10)$$

If a body is now placed in the jet, the Knudsen number would be based on the density of the Na atoms at the body and a characteristic length associated with the body. Since the density of sodium atoms decreases as the square of the distance from the orifice, very large Knudsen numbers can be obtained at distances of a few centimeters. In free molecular flow, the Knudsen number loses its meaning since the velocity distribution is not Maxwellian. However, it still gives one an idea about the geometry of the experiment as compared to the density of the gas.

## Experiment

A schematic diagram of the experimental equipment is shown in Fig. 1. The Na source shown in cross section consists of a $\frac{1}{8}$-in. wall stainless steel cylinder, $3\frac{1}{2}$ in. long and 1 in. diameter with a removable orifice plate containing an 0.0135-in.-diam hole. The heating coil consists of a Nichrome wire threaded through ceramic beads. An aluminum cylinder was placed over the source to form a radiation shield, and the ends were capped with soapstone disks. The entire source assembly was located inside the bell jar $B$. The orifice plate temperature was measured by an iron-constantan thermocouple. The light source consisted of an Osram lamp $S$ and collimating lens $L$, which were placed inside a light tight box $F$ located outside the bell jar. Radiation scattered from the Na jet was observed as right angles to the incident beam with a
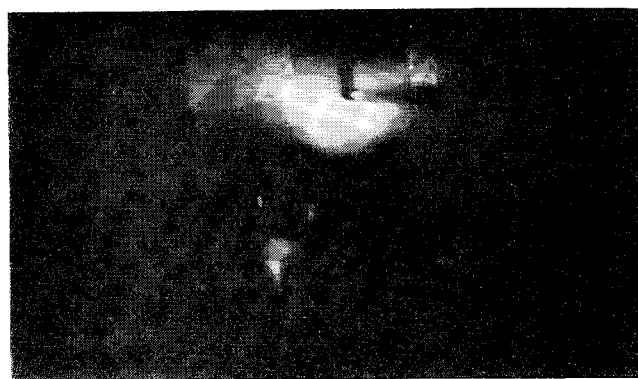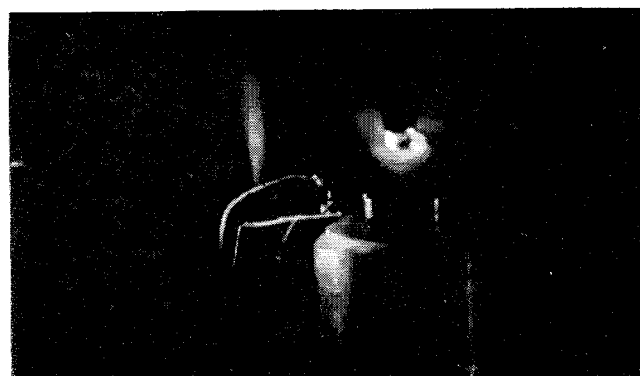
Polaroid Camera using Polaroid 3000 ASA film. The bell jar pressure was measured with a Phillips gage.

To operate the system, solid Na was placed inside the source. The system was evacuated to about $10^{-5}$ mm Hg. In order to obtain good outgassing of the vacuum system and source, the source was maintained at 200°C. The temperature then was raised to 300°C to vaporize the Na. By illuminating the region around the orifice, the scattering sodium atoms then were observed to form a free molecular jet. The aluminum plate $C$ served to condense the impinging Na atoms.

## Results

For this initial experiment, the objective was simply to observe a free molecular flow around two-dimensional body

configurations. Two bodies were chosen. The first was a glass cylinder approximately $\frac{1}{8}$ in. in diameter, and the second was a flat glass plate approximately $\frac{1}{2}$ in. wide. The models were mounted approximately $\frac{1}{2}$ in. above the orifice spanning the jet.

The free molecular jet effusion for a source temperature of 300°C is shown in Fig. 2a. The ambient pressure here was approximately $10^{-5}$ mm Hg, which was the limit of the present pumping system. The source pressure obtained from the known vapor pressure curve for Na was approximately $10^{-2}$ mm Hg. The number of molecules per unit time escaping the orifice was calculated to be approximately $10^{15}$ by means of Eq. (7). From Eqs. (8) and (9) one obtains a mean free path upstream of the orifice of 1.5 cm and a Knudsen number of 40. The number density inside the source may be calculated provided the vapor pressure of Na is known. The number density at the orifice was found to be about $10^{14}$ cm$^{-3}$, which is considerably higher than required for flow visualization by resonance scattering. At a distance of 4 cm from the orifice, the number density of the Na atoms was about $10^{10}$ cm$^{-3}$. In the present experiment, it was not possible to eliminate the reflections of the incident beam from the bell jar. These reflections appear as bright spots in the photograph.

Figure 2b is a photograph of a $\frac{1}{8}$-in.-diam glass cylinder in the jet. The source conditions are the same as those in the first photograph. The dark shadow running to the right from the cylinder is the light shadow cast by the cylinder in the beam of sodium light. The darker shadow above the model indicates the atomless wake cast by the model. The edges of the wake are straight lines that intersect at the orifice. This indicates that the particle path from the source is a straight line. The field of view in the photographs is limited by the width of the incident sodium light beam. However, during the course of the experiment the beam was moved to a distance of approximately 6 cm from the source. The jet was still visible at this distance where the number density is lower than $10^{10}$ cm$^{-3}$. It appears, therefore, that the resonance scattering technique may be used at number densities considerably less than $10^{10}$ cm$^{-3}$.

The free molecular flow over a plate $\frac{1}{2}$ in. wide is shown in Fig. 2c. The source conditions are the same as before. The characteristic straight line wake behind the model is again evident.

Several attempts were made to increase the source density sufficiently to observe the formation of a shock wave in front of the plate. A practical difficulty arose when this was attempted. To increase the density of Na in the source chamber, the temperature must be raised considerably. It was found that the condensation plate, which was at room temperature, did not condense the Na jet completely, and consequently the entire bell jar was fogged and the features of the jet were masked out. Another problem that arose with the present source design was that, at higher temperatures, some of the liquid Na passed through the orifice. When this happened, the bell jar fogged rapidly and masked out the jet.

## Conclusions

The results of this experiment show that the resonance scattering technique may be used to obtain good photographs in the free molecular flow regime. The technique has not yet been extended to the continuum and transitional regimes, but in principle there should be no great difficulty in photographing these regimes. The only requirement for use of the technique is to obtain number densities of the order of $10^{10}$ cm$^{-3}$. If the number densities are too high, multiple scattering will complicate the interpretation of the results. For this reason, it appears that to photograph higher density flow fields it may be desirable to seed the working gas with sodium to obtain visualization. It is clear that the same technique can be used not only in the case of sodium but in principle for all gases.

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# Heat Loss and Reaction Order Effects on Heat Release of Two-Stage Combustion Chambers

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HOMOGENEOUS chemical or "well-stirred" reactor theory has been applied in recent years to a number of engineering studies of chemical kinetics in high output combustion operations.[1–3] Although some authors mention the effects of heat loss[4–6] and reaction order[5,6] on the heat release of single stage combustion chambers, no one has presented a detailed theoretical analysis of such effects. Since current interest is directed towards multiple stage combustion chambers,[3,7] the results presented herein describe heat loss and reaction order effects on the specific heat release of a two-stage combustion chamber.

Following the approach of DeZubay,[7] the operation of a two-stage combustion chamber is described by a system of two single-stage homogeneous chemical reactors connected in series. The first stage receives a fresh charge of fuel and oxidant. Additional fresh oxidant and the partially burned products from the first stage are mixed and permitted to react further in the second stage. Complete details of the model and nomenclature used here are presented in Ref. 7. The first and second stages operate at stoichiometric and twice stoichiometric air-fuel ratios, respectively. Both stages have the same volume, inlet temperature, and inlet pressure. Propane and air are chosen as the fuel and oxidant.

### Heat Loss Effects

For the determination of heat loss effects, the overall reaction order $n$ is taken as 2.0, and the exponents of the concentration terms in the rate equation are assumed equal to 1.0. Heat loss is expressed as a percentage $B$ of the heat released. For convenience, $B$ of the first stage is assumed equal to $B$ of the second stage $(B_1 = B_2)$.‡

For the derivation of equations describing the two-stage system, the reader is referred to Ref. 7. To include heat loss effects, Eqs. (5) and (7) of Ref. 7 are modified to

$$y_1 = \frac{c(\varphi_1 + 1)(T_1 - T_0)}{H\psi_1(1 - B)} \tag{1}$$

$$y_2 = \frac{c[(\varphi_2 + 1)T_2 - (\varphi_2 - \varphi_1)T_0 - (\varphi_1 + 1)T_1]}{H\psi_2(1 - B)(1 - \psi_1 y_1)} \tag{2}$$